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Note

Latin Squares and Superqueens

F. K. HWANG

Bell Laboratories, Murray Hill, New Jersey

AND

KO-WEI LIH

Institute of Mathematics, Academia Sinica, Taipei, Taiwan, Republic of China

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Let L be a Latin square of order n with entries from $\{0, 1,..., n-1\}$. In addition, L is said to have the (n, k) property if, in each right or left wrap around diagonal, the number of cells with entries smaller than k is exactly k. It is established that a necessary and sufficient condition for the existence of Latin squares having the (n, k) property is that of $(2 \mid n \Rightarrow 2 \mid k)$ and $(3 \mid n \Rightarrow 3 \mid k)$. Also, these Latin squares are related to a problem of placing nonattacking queens on a toroidal chessboard.

An $n \times n$ array, $L = \{(i,j) \mid 0 \leqslant i, j < n\}$, is a Latin square of order n if its cells have entries from $\{0, 1,..., n-1\}$ in such a way that every number appears exactly once in each row and each column. For $0 \leqslant j < n$, the jth right diagonal of L is defined to be the set of cells $\{(i, j+i) \mid i = 0, 1,..., n-1\}$, and the jth left diagonal is the set $\{(i, j-i) \mid i = 0, 1,..., n-1\}$. (We adopt the convention that all artithmetic done in this paper is modulo n unless specified otherwise.) A diagonal is said to be complete if each of $0, 1,..., n-1$ appears in that diagonal. A Latin square L is called a Knut Vik design if every right and left diagonal of L is complete. A necessary and sufficient condition for the existence of Knut Vik designs, as established in Hedayat [2], is that n not be divisible by 2 or 3. The definition Knut Vik designs can be relaxed in various ways to obtain more comprehensive classes of interesting Latin squares. For example, Hwang [3] introduced the notion of a crisscross Latin square, which is a Latin square having all even right

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diagonals and all odd left diagonals complete. It is known from [3] that a necessary and sufficient condition for the existence of crisscross Latin squares is that n be divisible by 4. In this paper, we generalize Knut Vik designs in yet another direction and establish a necessary and sufficient condition for their existence.

We say that a Latin square L of order n has the (n, k) property, $1 \leqslant k \leqslant n$, if, on any right or left diagonal, the number of cells with entries less than k is exactly k. Clearly, a Latin square of order n is a Knut Vik design if and only if it has the (n, k) property simultaneously for $k = 1, 2, \ldots, n$.

We shall see that Latin squares with the (n, k) property are closely related to a classical problem concerning the chessboard. A queen on a chessboard can attack any other piece on the same horizontal, vertical, or ordinary diagonal lines. If we treat the chessboard as if it were a torus and allow a queen to wrap around to the opposite edge, then we call it a superqueen. Can we place n mutually nonattacking superqueens on an $n \times n$ chessboard? Pólya [4] shows that the answer is *yes* if and only if n is not divisible by 2 or 3. We say that the (n, k) superqueen problem is solvable if we can place nk superqueens on an $n \times n$ chessboard such that every superqueen attacks exactly $k-1$ other superqueens along each of the horizontal, vertical, right, and left diagonal lines through itself. Put in this terminology, Pólya's result means that the $(n, 1)$ superqueen problem is solvable if and only if n is not divisible by 2 or 3. The reader is referred to Chandra [1] for another independent generalization of Pólya's result.

THEOREM 1. *Let* $1 \leqslant k \leqslant n$. *The following statements are equivalent*:

(i) *There exists a Latin square having the* (n, k) *property.*

(ii) *The* (n, k) *superqueen problem is solvable.*

(iii) *Conditions* $(2 \mid n \Rightarrow 2 \mid k)$ *and* $(3 \mid n \Rightarrow 3 \mid k)$ *apply.*

Proof. (i) $\Rightarrow$ (ii). This is obvious when we place superqueens on cells whose entries are less than k.

(ii) $\Rightarrow$ (iii). Suppose the (n, k) superqueen problem is solvable. Consider the cells occupied by superqueens, $(x_0, y_0), (x_1, y_1), \ldots, (x_{nk-1}, y_{nk-1})$. Let N_k denote the mutiset

$$\{\underbrace{0, \ldots, 0}_{k},\ \underbrace{1, \ldots, 1}_{k},\ \ldots,\ \underbrace{n-1, \ldots, n-1}_{k}\}.$$

Then, we have $N_k = \{x_i\} = \{y_i\} = \{y_i + x_i\} = \{y_i - x_i\}$, where $i = 0, 1, \ldots, nk-1$ in each of the four sets. It follows that

$$k \sum_{i=0}^{n-1} i = \sum_{i=0}^{nk-1} (y_i - x_i) = \sum_{i=0}^{nk-1} y_i - \sum_{i=0}^{nk-1} x_i = 0,$$

i.e., n divides $kn(n-1)/2$. Thus, k is even if n is. On the otherhand, we also have

$$\begin{aligned}
k\sum_{i=0}^{n-1} i^2 &= \sum_{i=0}^{nk-1} (y_i - x_i)^2 \\
&= \sum_{i=0}^{nk-1} y_i^2 + \sum_{i=0}^{nk-1} x_i^2 - 2\sum_{i=0}^{nk-1} x_i y_i \\
&= 2k\sum_{i=0}^{n-1} i^2 - 2\sum_{i=0}^{nk-1} x_i y_i,
\end{aligned}$$

i.e., $k\sum_{i=0}^{n-1} i^2 = 2\sum_{i=0}^{nk-1} x_i y_i$. We know, however, that

$$\sum_{i=0}^{nk-1} (y_i + x_i)^2 - \sum_{i=0}^{nk-1} (y_i - x_i)^2 = 4\sum_{i=0}^{nk-1} x_i y_i = 0.$$

So n divides $kn(n-1)(2n-1)/3$. Hence k is divisible by 3 if n is.

(iii) $\Rightarrow$ (i). This part will follow from Theorem 2.

THEOREM 2. *There exists a Latin square $L_n = (l_{ij})$ with the (n, k) property simultaneously for*

(i) $k = 1, 2, 3, \ldots, n$ *if* $2 \nmid n$ *and* $3 \nmid n$,

(ii) $k = 2, 4, 6, \ldots, n$ *if* $2 \mid n$ and $3 \nmid n$,

(iii) $k = 3, 6, 9, \ldots, n$ *if* $2 \nmid n$ *and* $3 \mid n$,

(iv) $k = 6, 12, 18, \ldots, n$ *if* $2 \mid n$ *and* $3 \mid n$.

Proof. Latin square L_n for (i) and (iii) is given in Case I. L_n for (ii) and (iv) is given in Cases II and III according to whether 4 divides n.

Case I. $2 \nmid n$. Let $l_{ij} = 2i + j$. Then, the entries on the bth left diagonal form the set $\{x + b \mid x = 0, 1, \ldots, n-1\}$ and the entries on the bth right diagonal form the set $\{3x + b \mid x = 0, 1, \ldots, n-1\}$. Clearly, L_n satisfies either (i) or (iii).

Case II. $2 \mid n$ but $4 \nmid n$. Let $l_{ij} = 5i + j$. The entries on the bth left diagonal form the set $\{4x + b \mid x = 0, 1, \ldots, n-1\}$ and the entries on the bth right diagonal form the set $\{6x + b \mid x = 0, 1, \ldots, n-1\}$. Again, it is easily verified that L_n satisfies either (ii) or (iv).

Case III. $4 \mid n$. We first construct a Latin rectangle $R_n = \{(i,j) \mid 0 \leqslant i < \frac{1}{2}n,\ 0 \leqslant j < n\}$ with entries r_{ij} such that

$$\begin{aligned}
r_{ij} &= 2i + j - 1, && \text{if } j \text{ is even,} \\
&= -2i + j - 1, && \text{if } j \text{ is odd.}
\end{aligned}$$

TABLE I

	bth right diagonal		bth left diagonal	
	b even	b odd	b even	b odd
Even subset	$6x+(b-1)$	$6x+(b+2)$	$2x+(b-1)$	$2x+b$
Odd subset	$-2x+(b-2)$	$-2x+(b-1)$	$-6x+(b-4)$	$-6x+(b-1)$

Each row of R_n is complete and each column has distinct entries. For $0 \leqslant b < n$, define the bth right diagonal of R_n to be the set of cells $\{(t, t+b) \mid t = 0, 1, \ldots, n-1\}$, where the first coordinate is reduced modulo $n/2$ and the second coordinate is reduced modulo n. The bth left diagonal is defined similarly.

Claim. If $3 \nmid n$, then each right or left diagonal of R_n is complete. If $3 \mid n$, then each right and left diagonal of R_n has exactly k entries less than k simultaneously for $k = 6, 12, 18, \ldots, n$.

To prove this, we partition each diagonal into two subsets according to whether the second coordinate of a cell is even or odd. Once the labels of even and odd subsets are clearly identified, the claim can be easily checked. The labels are classified in Table I. In each of the listed cases, $x = 0, 1, \ldots, n/2 - 1$.

Now, let R'_n be the $\frac{1}{2}n \times n$ Latin rectangle obtained from R_n by adding 1 to every even entry and subtracting 1 from every odd entry. Finally, define $L_n = \left[\begin{smallmatrix} R_n \\ R'_n \end{smallmatrix}\right]$. It is straightforward to verify (ii) and (iv) for L_n.

The Latin squares of the three smallest orders constructed by the method in Case III are displayed below to help the reader visualize the procedure.

$$L_4:\quad \begin{matrix} 3 & 0 & 1 & 2 \\ 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 3 \\ 0 & 3 & 2 & 1 \end{matrix}$$

$$L_8:\quad \begin{matrix} 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 3 & 0 & 5 & 2 & 7 & 4 \\ 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 \\ 5 & 2 & 7 & 4 & 1 & 6 & 3 & 0 \\ 6 & 1 & 0 & 3 & 2 & 5 & 4 & 7 \\ 0 & 7 & 2 & 1 & 4 & 3 & 6 & 5 \\ 2 & 5 & 4 & 7 & 6 & 1 & 0 & 3 \\ 4 & 3 & 6 & 5 & 0 & 7 & 2 & 1 \end{matrix}$$

L_{12}:	11	0	1	2	3	4	5	6	7	8	9	10
	1	10	3	0	5	2	7	4	9	6	11	8
	3	8	5	10	7	0	9	2	11	4	1	6
	5	6	7	8	9	10	10	0	1	2	3	4
	7	4	9	6	11	8	1	10	3	0	5	2
	9	2	11	4	1	6	3	8	5	10	7	0
	10	1	0	3	2	5	4	7	6	9	8	11
	0	11	2	1	4	3	6	5	8	7	10	9
	2	9	4	11	6	1	8	3	10	5	0	7
	4	7	6	9	8	11	10	1	0	3	2	5
	6	5	8	7	10	9	0	11	2	1	4	3
	8	3	10	5	0	7	2	9	4	11	6	1

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